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Kloosterman's uniformly distributed sequence ☆

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Abstract

Applying the theory of uniform distribution, especially the Erdös–Turán–Koksma inequality and the Koksma–Hlawka inequality, to the two-dimensional Kloosterman sequence $(a_j/n, a_j^*/n)$, $j = 1, 2, \ldots, \varphi(n)$ (where $a_j a_j^* \equiv 1 \pmod n$, $a_j, a_j^* \in [1, n]$ and $\varphi(n)$ is the Euler function) we find an estimation for the discrepancy $D^*_{\varphi(n)}$ of this sequence and an error term for the Kth moment, $K = 1, 2, \ldots,$ of the sequence of distances $|a_j/n - a_j^*/n|$ as

$$\left|\frac{1}{\varphi(n)}\sum_{j=1}^{\varphi(n)}\left|\frac{a_j}{n}-\frac{a_j^*}{n}\right|^K - \int_0^1\int_0^1 |x-y|^K \,\mathrm{d}x\,\mathrm{d}y\right| \leqslant V(|x-y|^K)D^*_{\varphi(n)},$$

where the Hardy–Krause variation $V(|x-y|^K) = 4$ and the discrepancy

$$D^*_{\varphi(n)}((a_j/n, a_j^*/n)) = O(d(n)\sqrt{n}\,(\log\varphi(n))^2/\varphi(n))$$

($d(n)$ is the divisor function). From known estimates of Kloosterman's sum immediately follows uniform distribution of the sequence $(a_j/n, a_j^*/n)$, $j = 1, 2, \ldots, \varphi(n)$, as $n \to \infty$ which directly implies that the related sequence of distances $|a_j/n - a_j^*/n|$ has the asymptotic distribution function $g(x) = 2x - x^2$. For a general sequence of points (x_j, y_j), $j = 1, 2, \ldots, N$, in the unit square $[0, 1)^2$ we find an approximation of the discrepancy of $|x_j - y_j|$ by the discrepancy of (x_j, y_j) which gives

$$D^*_{\varphi(n)}(|a_j/n - a_j^*/n|) = O((d(n))^2\sqrt{n}\,(\log\varphi(n))^2/\varphi(n)).$$

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These results improve and unify some of Zhang's results published in Zhang (J. Number Theory **52** (1995) 1–6; J. Number Theory **61** (1996) 301–310, Acta Math. Hungar. **76** (1997) 17–30) from the point of view of uniform distribution theory.

1. Introduction

Let $n \geqslant 2$ be an integer. Let $a_1 = 1 < a_2 < \cdots < a_{\varphi(n)} < n$, be the sequence of all integers in $[1, n]$ coprime to n, where φ denotes the Euler function. Define a_j^*, $0 < a_j^* < n$, by the congruence $a_j a_j^* \equiv 1 \pmod n$. Then the two-dimensional sequence of blocks A_n, $n = 2, 3, \ldots,$

$$A_n = \left(\left(\frac{a_1}{n}, \frac{a_1^*}{n} \right), \left(\frac{a_2}{n}, \frac{a_2^*}{n} \right), \ldots, \left(\frac{a_{\varphi(n)}}{n}, \frac{a_{\varphi(n)}^*}{n} \right) \right),$$

is uniformly distributed (abbreviating u.d.). This follows directly from Weyl's criterion and from the classical bound of the Kloosterman sum

$$\left| \sum_{j=1}^{\varphi(n)} e^{2\pi i \left(a \frac{a_j}{n} + b \frac{a_j^*}{n} \right)} \right| \leqslant \sqrt{(a, b, n)} d(n) \sqrt{n} \tag{1}$$

for any integers a, b and $n \geqslant 2$, where $d(n)$ is the divisor function and (a, b, n) is the greatest common divisor (cf. [E]).

Using A_n define a one-dimensional sequence of blocks B_n, $n = 2, 3, \ldots,$

$$B_n = \left(\left| \frac{a_1}{n} - \frac{a_1^*}{n} \right|, \left| \frac{a_2}{n} - \frac{a_2^*}{n} \right|, \ldots, \left| \frac{a_{\varphi(n)}}{n} - \frac{a_{\varphi(n)}^*}{n} \right| \right).$$

Zhang [Z1] formulated the distribution of B_n as an open problem. But from u.d. of A_n it follows directly that B_n has the asymptotic distribution function (abbreviating a.d.f.)

$$g(x) = 2x - x^2,$$

since

$$\int\int_{\substack{|u-v|<x \\ (u,v)\in[0,1]^2}} 1 \, \mathrm{d}u \, \mathrm{d}v = 2x - x^2.$$

Zhang [Z1] also found the even moments

$$\frac{1}{\varphi(n)} \sum_{i=1}^{\varphi(n)} \left| \frac{a_i}{n} - \frac{a_i^*}{n} \right|^{2k} = \frac{1}{(2k+1)(k+1)} + O\left(4^k \frac{d^2(n)\sqrt{n}}{\varphi(n)} (\log n)^2 \right),$$

where $\varphi(n)$ is the Euler function, $d(n)$ is the divisor function and the O provides an absolute constant. For an odd moment $2k+1$ the leading term is again

$$\int_0^1 x^{2k+1}\mathrm{d}(2x-x^2)=\frac{1}{(2k+3)(k+1)}.$$

Zhang (1997) found this in [Z3] without the factor 4^k in the O-term.

Applying some parts of the uniform distribution (u.d.) theory we find, for every Kth moment, $K=1,2,\dots$, the error term

$$O\left(\frac{d(n)\sqrt{n}}{\varphi(n)}(\log\varphi(n))^2\right).$$

Zhang [Z2] also found the a.d.f. $g(x)=2x-x^2$ of B_n with the star-discrepancy

$$O\left(\frac{d^2(n)\sqrt{n}}{\varphi(n)}(\log n)^3\right).$$

In [Z3, (1997)], he improved this estimation finding $(\log n)^2$ instead of $(\log n)^3$.

For definitions and basic results in u.d. theory we use the monograph by Kuipers and Niederreiter [KN] or the more recent by Drmota and Tichy [DT]. The paper has the following plan:

We start with the so-called Koksma–Hlawka inequality (cf. [DT, Theorem 1.14, p. 10]; [KN, Theorem 5.5, p. 151]). Let (x_j,y_j), $j=1,2,\dots,N$, be a finite sequence of points in the unit square $[0,1]^2$. The Koksma–Hlawka inequality for dimension 2 gives

$$\left|\frac{1}{N}\sum_{j=1}^{N}F(x_j,y_j)-\int_0^1\int_0^1 F(x,y)\,\mathrm{d}x\,\mathrm{d}y\right|\leqslant V(F(x,y))D_N^*((x_j,y_j)),\tag{2}$$

where $V(F(x,y))$ is the Hardy–Krause variation of the function $F(x,y)$ and $D_N^*((x_j,y_j))$ is the so-called star discrepancy of $(x_1,y_1),\dots,(x_N,y_N)$ defined as

$$D_N^*((x_j,y_j))=\sup_{x,y\in[0,1]}\left|\frac{\#\{j\leqslant N;(x_j,y_j)\in[0,x)\times[0,y)\}}{N}-xy\right|.$$

To approximate of the Kth moment we put $F(x,y)=|x-y|^K$ and in Part 2 we prove that

$$V(|x-y|^K)=4.$$

We replace the star discrepancy $D_N^*((x_j, y_j))$ in (2) by the so-called extremal discrepancy $D_N((x_j, y_j))$ defined as

$$D_N((x_j, y_j)) = \sup_{\substack{x,y,x',y' \in [0,1] \\ x \leqslant x', y \leqslant y'}} \left| \frac{\#\{j \leqslant N; (x_j, y_j) \in [x, x') \times [y, y')\}}{N} - (x' - x)(y' - y) \right|$$

and then we use the Erdös–Turán–Koksma inequality (cf. [DT, DT, Theorem 1.21, p. 15], [KN, p. 116]) reduced to the dimension 2:

$$D_N((x_j, y_j)) \leqslant \left(\frac{3}{2}\right)^2 \left(\frac{2}{H+1} + \sum_{0 < ||h|| \leqslant H} \frac{1}{r(\mathbf{h})} \left| \frac{1}{N} \sum_{j=1}^{N} e^{2\pi i (h_1 x_j + h_2 y_j)} \right| \right), \tag{3}$$

where $r(\mathbf{h}) = \max(1, |h_1|)\max(1, |h_2|)$, $||\mathbf{h}|| = \max(|h_1|, |h_2|)$ for $\mathbf{h} = (h_1, h_2) \in \mathbf{Z}^2$ and H is an arbitrary positive integer. Putting $N = \varphi(n)$, $(x_j, y_j) = (a_j/n, a_j^*/n)$, in Part 3 we find the extremal discrepancy of A_n

$$D_{\varphi(n)}\left(\left(\frac{a_j}{n}, \frac{a_j^*}{n}\right)\right) = O\left(\frac{d(n)\sqrt{n}}{\varphi(n)} (\log \varphi(n))^2\right).$$

In Part 4 we derive a boundary of the star discrepancy of the one-dimensional difference sequence

$$|x_1 - y_1|, \ldots, |x_N - y_N|$$

with respect to d.f. $g(x) = 2x - x^2$ from the star discrepancy of the two-dimensional sequence

$$(x_1, y_1), \ldots, (x_N, y_N),$$

for an arbitrary sequence (x_j, y_j), $j = 1, 2, \ldots, N$ in the unit square $[0, 1]^2$. We prove

$$D_N^*(|x_j - y_j|) \leqslant 4\sqrt{D_N^*((x_j, y_j))}.$$

The constant 4 is better than the constants $8\sqrt{2} + 1$ or 8 resulting from inequalities on isotropic discrepancy in [KN, Theorem 1.6, p.95] or in [N, p. 17], respectively.

Analyzing [Z2] and assuming that the sequence (x_j, y_j), $j = 1, 2, \ldots, N$ is invariant under transformations $(x, y) \to (y, x)$ and $(x, y) \to (1 - x, 1 - y)$ we improve this, in Part 5, to

$$D_N^*(|x_j - y_j|) \leqslant 3D_N((x_j, \{y_j - x_j\})) + D_N((x_j, y_j)),$$

where $\{x\}$ is the fractional part of x. Applying this to the sequence $(x_j, y_j) = (a_j/n, a_j^*/n)$ in Part 6 we find

$$D_{\varphi(n)}\left(\left|\frac{a_j}{n} - \frac{a_j^*}{n}\right|\right) = O\left(\frac{d^2(n)\sqrt{n}}{\varphi(n)} (\log \varphi(n))^2\right)$$

which is a little better than in [Z3], where $\log \varphi(n)$ is replaced by $\log n$.

2. Hardy–Krause variation of $|x-y|^K$

For a function $F(x,y)$ of two variables, the Hardy–Krause variation $V(F)$ is defined as

$$V(F(x,y)) = V^{(1)}(F(x,1)) + V^{(1)}(F(1,y)) + V^{(2)}(F(x,y)), \tag{4}$$

where $V^{(i)}$ are the classical Vitali's variations, i.e.

$$\begin{aligned} V^{(1)}(F(x,1)) &= \sup_P \sum_{i=0}^{k-1} |F(x'_{i+1},1) - F(x'_i,1)|, \\ V^{(2)}(F(x,y)) &= \sup_P \sum_{i=0}^{k-1} \sum_{j=0}^{s-1} |F(x'_{i+1},y'_{j+1}) - F(x'_i,y'_{j+1}) \\ &\quad - F(x'_{i+1},y'_j) + F(x'_i,y'_j)|, \end{aligned} \tag{5}$$

where the supremum is taken over all partitions P of $[0,1]^2$ of the form

$$P = \{[x'_i, x'_{i+1}] \times [y'_j, y'_{j+1}] \quad i = 0, \dots, k-1 \quad j = 0, \dots, s-1\},$$

where

$$0 = x'_0 < x'_1 < \dots < x'_k = 1 \quad \text{and} \quad 0 = y'_0 < y'_1 < \dots < y'_s = 1$$

are partitions of $[0,1]$ in x- and y-axis, respectively. Without loss of generality, we can assume that these partitions coincide.

For $F(x,y) = |x-y|^K$, $K = 1, 2, \dots$, we have

$$V^{(1)}((1-x)^K)) = \int_0^1 \left| \frac{\mathrm{d}(1-x)^K}{\mathrm{d}x} \right| \mathrm{d}x = 1.$$

For computing $V^{(2)}(|x-y|^K))$, we divide any partition P to three parts:

P_1 contains rectangles lying above the diagonal $y = x$;
P_2 contains rectangles lying under the diagonal $y = x$; and
P_3 contains rectangles having diagonals on the line $y = x$.

We denote the supremum of the sum in definition (5) of $V^{(2)}(|x-y|^K)$ over P_i by $V_i^{(2)}$. Directly

$$V_1^{(2)} = \int_0^1 \mathrm{d}x \int_x^1 \left| \frac{\partial^2 (y-x)^K}{\partial x \partial y} \right| \mathrm{d}y = \begin{cases} 1 & \text{for } K = 2, 3, \dots, \\ 0 & \text{for } K = 1. \end{cases}$$

The same holds for $V_2^{(2)}$.

Now, we evaluate $V_3^{(2)}$. Let Q be the number of squares in P_3 and let $t>0$ be sufficiently small. Then we extend the partition P to a new partition P such that all terms in P_3 are of the form $[x_i', x_i'+t] \times [x_i', x_i'+t]$ except at most Q squares $[x_i', x_i'+t'] \times [x_i', x_i'+t']$ with some variable $t'<t$. Since all rectangles in any partition are disjoint, the number of $[x_i', x_i'+t] \times [x_i', x_i'+t]$ is bounded by $\frac{\sqrt{2}}{\sqrt{2}t}$ and moreover

$$||x_{i+1}'-y_{j+1}'|^K - |x_i'-y_{j+1}'|^K - |x_{i+1}'-y_j'|^K + |x_i'-y_j'|^K| = 2t^K.$$

Thus

$$V_3^{(2)} \leqslant \frac{\sqrt{2}}{\sqrt{2}t} 2t^K + Q2t^K,$$

and, as $t\to 0$, and for $K=2,3,\dots$, we have the limit $V_3^{(2)}=0$. For $K=1$ we can find partitions P'' such that this limit is equal $V_3^{(2)}=2$. Thus,

$$V^{(2)}(|x-y|^K) = \begin{cases} 1+1+0=2 & \text{for } K=2,3,\dots, \\ 0+0+2=2 & \text{for } K=1 \end{cases}$$

and for the Hardy–Krause variation (4) we have

$$V(|x-y|^K) = 1+1+2=4.$$

This gives

$$\left| \frac{1}{N} \sum_{j=1}^{N} |x_j-y_j|^K - \frac{2}{(K+1)(K+2)} \right| \leqslant 4.D_N^*((x_j,y_j))$$

for every sequence $(x_1,y_1),\dots,(x_N,y_N)$ in $[0,1]^2$.

3. Discrepancy of A_n

In the case

$$(x_j,y_j) = \left(\frac{a_j}{n}, \frac{a_j^*}{n}\right), \quad N=\varphi(n),$$

and for an arbitrary positive integer H, the two-dimensional Erdös–Turán–Koksma inequality (3) takes the form

$$\begin{aligned} D_{\varphi(n)}\left(\left(\frac{a_j}{n}, \frac{a_j^*}{n}\right)\right) \leqslant \left(\frac{3}{2}\right)^2 \Bigg(& \frac{2}{H+1} + \sum_{h_1,h_2=1}^{H} \frac{2}{h_1h_2} \left| \frac{1}{\varphi(n)} \sum_{j=1}^{\varphi(n)} e^{2\pi i\left(h_1\frac{a_j}{n}+h_2\frac{a_j^*}{n}\right)} \right| \\ & + \sum_{h_1,h_2=1}^{H} \frac{2}{h_1h_2} \left| \frac{1}{\varphi(n)} \sum_{j=1}^{\varphi(n)} e^{2\pi i\left(-h_1\frac{a_j}{n}+h_2\frac{a_j^*}{n}\right)} \right| \\ & + \sum_{h_1=1}^{H} \frac{4}{h_1} \left| \frac{1}{\varphi(n)} \sum_{j=1}^{\varphi(n)} e^{2\pi i h_1\frac{a_j}{n}} \right| \Bigg). \end{aligned} \tag{6}$$

For the first two sums we use (1), and since

$$\sum_{h_1,h_2=1}^{H} \frac{\sqrt{(h_1,h_2,n)}}{h_1h_2} \leqslant \sum_{d=1}^{\infty} \frac{\sqrt{d}}{d^2} \sum_{i_1,i_2=1}^{H} \frac{1}{i_1 i_2} \leqslant 3\,(\log(eH))^2,$$

we obtain the upper bound

$$12\,\frac{d(n)\sqrt{n}}{\varphi(n)}\,(\log(eH))^2.$$

For the final sum in (6) we use the Ramanujan identity

$$\sum_{j=1}^{\varphi(n)} e^{2\pi i h\frac{a_j}{n}} = \frac{\varphi(n)}{\varphi\left(\frac{n}{(h,n)}\right)}\mu\left(\frac{n}{(h,n)}\right)$$

and

$$\sum_{h_1=1}^{H} \frac{1}{h_1\varphi\left(\frac{n}{(h_1,n)}\right)} \leqslant \sum_{d|n} \frac{1}{d\varphi\left(\frac{n}{d}\right)} \sum_{h=1}^{H} \frac{1}{h} = \frac{1}{n} \sum_{d|n} \frac{d}{\varphi(d)} \sum_{h=1}^{H} \frac{1}{h} \leqslant \frac{d(n)}{\varphi(n)} \log(eH),$$

where, for $d|n$ we use $d/\varphi(d) \leqslant n/\varphi(n)$. Summing up these bounds we have

$$\begin{aligned} &D_{\varphi(n)}\left(\left(\frac{a_j}{n},\frac{a_j^*}{n}\right)\right) \\ &\leqslant \left(\frac{3}{2}\right)^2 \left(\frac{2}{H+1} + 12\frac{d(n)\sqrt{n}}{\varphi(n)}\,(\log(eH))^2 + 4\,\frac{d(n)}{\varphi(n)}\log(eH)\right), \end{aligned}$$

which for $eH = \varphi(n)$ gives

$$D_{\varphi(n)}\left(\left(\frac{a_j}{n},\frac{a_j^*}{n}\right)\right) \leqslant \left(\frac{3}{2}\right)^2 17\,\frac{d(n)\sqrt{n}}{\varphi(n)}\,(\log\varphi(n))^2 \tag{7}$$

for $n \geqslant 8$.

4. Star discrepancies of $|x_j - y_j|$ and (x_j, y_j)

Again, applying the Koksma–Hlawka inequality, we prove the following claim.

Claim. *For an arbitrary* $N = 1, 2, \ldots,$ *and for any two finite sequences* $x_1, \ldots, x_N$ *and* $y_1, \ldots, y_N$ *in* $[0, 1]$,

$$D_N^*(|x_j - y_j|) \leqslant 4\sqrt{D_N^*((x_j, y_j))}, \tag{8}$$

where $D_N^*(|x_j - y_j|)$ is the star discrepancy of $|x_1 - y_1|, \dots, |x_N - y_N|$ with respect to d.f. $g(x) = 2x - x^2$, i.e.

$$D_N^*(|x_j - y_j|) = \sup_{x\in[0,1]} \left| \frac{\#\{j \leqslant N; |x_j - y_j| \in [0, x)\}}{N} - g(x) \right|.$$

Proof. Put $z_j = |x_j - y_j|$, let $c_I(x)$ be the indicator function of the interval I, $g(x) = 2x - x^2$, and define the following auxiliary functions: For any $\varepsilon > 0$ and $x_0 \in (0, 1)$, let

$$f_1(x) = \begin{cases} 1 & \text{for } x \in [0, x_0), \\ 1 - (x - x_0)\dfrac{1}{\varepsilon} & \text{for } x \in [x_0, \min(1, x_0 + \varepsilon)), \\ 0 & \text{for } x \in [\min(1, x_0 + \varepsilon), 1], \end{cases}$$

$$f_2(x) = \begin{cases} 1 & \text{for } x \in [0, \max(0, x_0 - \varepsilon)), \\ 1 - (x - (x_0 - \varepsilon))\frac{1}{\varepsilon} & \text{for } x \in [\max(0, x_0 - \varepsilon), x_0), \\ 0 & \text{for } x \in [x_0, 1]. \end{cases}$$

If $\frac{1}{N}\sum_{j=1}^N c_{[0,x_0)}(z_j) \geqslant \int_0^{x_0} 1.\mathrm{d}g(x)$, then

$$\frac{1}{N}\sum_{j=1}^N c_{[0,x_0)}(z_j) - \int_0^{x_0} 1.\mathrm{d}g(x) \leqslant \left| \frac{1}{N}\sum_{j=1}^N f_1(z_j) - \int_0^1 f_1(x)\,\mathrm{d}g(x) \right| + \int_{x_0}^{\min(1,x_0+\varepsilon)} f_1(x)\,\mathrm{d}g(x).$$

If $\frac{1}{N}\sum_{j=1}^N c_{[0,x_0)}(z_j) \leqslant \int_0^{x_0} 1.\mathrm{d}g(x)$, then

$$\int_0^{x_0} 1.\mathrm{d}g(x) - \frac{1}{N}\sum_{j=1}^N c_{[0,x_0)}(z_j) \leqslant \left| \frac{1}{N}\sum_{j=1}^N f_2(z_j) - \int_0^1 f_2(x)\,\mathrm{d}g(x) \right| + \int_{\max(0,x_0-\varepsilon)}^{x_0} (1 - f_2(x))\,\mathrm{d}g(x).$$

Clearly,

$$\int_{x_0}^{\min(1,x_0+\varepsilon)} f_1(x)\,\mathrm{d}g(x) \leqslant \varepsilon \text{ and } \int_{\max(0,x_0-\varepsilon)}^{x_0} (1 - f_2(x))\,\mathrm{d}g(x) \leqslant \varepsilon.$$

Putting $F_i(x,y) = f_i(|x-y|)$, $i=1,2$, and since

$$\frac{1}{N}\sum_{j=1}^{N} f_i(z_j) - \int_0^1 f_i(x)\,\mathrm{d}g(x) = \frac{1}{N}\sum_{j=1}^{N} F_i(x_j,y_j) - \int_0^1\int_0^1 F_i(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

applying the Koksma–Hlawka inequality (2), we find

$$\left|\frac{1}{N}\sum_{j=1}^{N} c_{[0,x_0)}(z_j) - \int_0^{x_0} 1.\mathrm{d}g(x)\right| \leqslant \max(V(F_1(x,y)), V(F_2(x,y))).D_N^*((x_j,y_j)) + \varepsilon,$$

for any $N = 1,2,\dots,$ $x_0 \in [0,1]$ and $\varepsilon > 0$. Now, we compute the Hardy–Krause variation of $F_i(x,y)$, $i=1,2$. Directly from graphs of $F_i(x,1)$ we find the one-dimensional Vitali's variations

$$V^{(1)}(F_1(x,1)) = \begin{cases} 1 & \text{for } x_0+\varepsilon \leqslant 1,\\ \dfrac{1-x_0}{\varepsilon} & \text{for } x_0+\varepsilon > 1,\end{cases}$$

$$V^{(1)}(F_2(x,1)) = \begin{cases} 1 & \text{for } x_0-\varepsilon \geqslant 0,\\ 1-\dfrac{\varepsilon-x_0}{\varepsilon} & \text{for } x_0-\varepsilon < 0,\end{cases}$$

and the same hold for $F_i(1,y)$. For two-dimensional Vitali's variation of $F_i(x,y)$, let P_0 be a partition of $[0,1]^2$ defined by a cartesian product of the one-dimensional partition

$$0 = x_0' < x_1' < \cdots < x_s' = 1 \tag{9}$$

of $[0,1]$. We consider the following refinement: Let $t > 0$ be a sufficiently small real number and assume that $\frac{x_0}{t}$ and $\frac{x_0+\varepsilon}{t}$ are integers. Thus $\frac{x_0}{\varepsilon}$ must be rational. To the partitions (9) we added all points

$$t < 2t < \cdots < t\left[\frac{1}{t}\right]$$

and we call P the cartesian product of the resulting partition of $[0,1]$. Thus, almost all rectangles (except at most s^2 terms) $[x_i', x_{i+1}'] \times [y_j', y_{j+1}']$ in P can be divided into the following subsets:

P_1 contains rectangles lying above the line $y = x + x_0 + \varepsilon$;
P_2 contains rectangles having diagonals on the line $y = x + x_0 + \varepsilon$;
P_3 contains rectangles lying between lines $y = x + x_0 + \varepsilon$ and $y = x + x_0$;
P_4 contains rectangles having diagonals on the line $y = x + x_0$;
P_5 contains rectangles lying between lines $y = x + x_0$ and $y = x$;
P_6 contains rectangles having diagonals on the line $y = x$;
P_7 contains rectangles lying between lines $y = x$ and $y = x - x_0$;
P_8 contains rectangles having diagonals on the line $y = x - x_0$;

P_9 contains rectangles lying between lines $y = x - x_0$ and $y = x - x_0 - \varepsilon$;
P_{10} contains rectangles having diagonals on the line $y = x - x_0 - \varepsilon$;
P_{11} contains rectangles lying under the line $y = x - x_0 - \varepsilon$.

We denote by $V_l^{(2)}(F_k)$ the supremum of the sum of

$$\left| F_k(x'_{i+1}, y'_{j+1}) - F_k(x'_i, y'_{j+1}) - F_k(x'_{i+1}, y'_j) + F_k(x'_i, y'_j) \right|$$

as $t \to 0$ and over the rectangles in P_l. Directly by computation it follows:

(I) For $x_0 + \varepsilon \leqslant 1$ we have nonzero $V_l^{(2)}(F_1)$ only for $l = 2, 4, 8, 10$ and $V_2^{(2)} = V_{10}^{(2)} = \frac{1-x_0-\varepsilon}{\varepsilon}$, $V_4^{(2)} = V_8^{(2)} = \frac{1-x_0}{\varepsilon}$.

(II) For $x_0 + \varepsilon > 1$ we have nonzero $V_l^{(2)}(F_1)$ only for $l = 4, 8$ and $V_4^{(2)} = V_8^{(2)} = \frac{1-x_0}{\varepsilon}$.

(III) For $x_0 - \varepsilon \geqslant 0$ and $V_l^{(2)}(F_2)$ we put $x_0 := x_0 - \varepsilon$ in (I).

(IV) For $x_0 - \varepsilon < 0$ we have nonzero $V_l^{(2)}(F_2)$ only for $l = 4, 6, 8$ and $V_4^{(2)} = V_8^{(2)} = \frac{1-x_0}{\varepsilon}$, $V_6^{(2)} = \frac{2}{\varepsilon}$.

Thus for the Hardy–Krause variation

$$V(F_k(x,y)) = V^{(1)}(F_k(x,y)) + V^{(1)}(F_k(x,y)) + V^{(2)}(F_k(x,y)),$$

we have

$$V(F_1(x,y)) = \begin{cases} 2 + 2\dfrac{1 - x_0 - \varepsilon}{\varepsilon} + 2\dfrac{1 - x_0}{\varepsilon} = 4\dfrac{1 - x_0}{\varepsilon} & \text{for } x_0 + \varepsilon \leqslant 1, \\ 2\dfrac{1 - x_0}{\varepsilon} + 2\dfrac{1 - x_0}{\varepsilon} = 4\dfrac{1 - x_0}{\varepsilon} & \text{for } x_0 + \varepsilon > 1, \end{cases}$$

$$V(F_2(x,y)) = \begin{cases} 2 + 2\dfrac{1 - (x_0 - \varepsilon) - \varepsilon}{\varepsilon} + 2\dfrac{1 - (x_0 - \varepsilon)}{\varepsilon} = 4\dfrac{1 - (x_0 - \varepsilon)}{\varepsilon} & \text{for } x_0 - \varepsilon \geqslant 0, \\ 2\left(1 - \dfrac{\varepsilon - x_0}{\varepsilon}\right) + 2\dfrac{1 - x_0}{\varepsilon} + \dfrac{2}{\varepsilon} = \dfrac{4}{\varepsilon} & \text{for } x_0 - \varepsilon < 0. \end{cases}$$

This implies

$$D_N^*(|x_j - y_j|) \leqslant \frac{4}{\varepsilon} D_N^*((x_j, y_j)) + \varepsilon$$

for any $\varepsilon > 0$, since the rationality of $\frac{x_0}{\varepsilon}$ can be omitted. To find (8) we put

$$\varepsilon = 2\sqrt{D_N^*((x_j, y_j))}. \qquad \square$$

5. Star discrepancy of $|x_j - y_j|$ for (x_j, y_j) invariant to some mappings

Claim. *If* $(x_1, y_1), \dots, (x_N, y_N)$ *is a sequence in* $[0,1)^2$ *invariant to* $(x,y) \to (y,x)$ *and* $(x,y) \to (1-x, 1-y)$, *i.e. for any* $i \leqslant N$ *there exists* $j_1, j_2 \leqslant N$ *such that* $(x_{j_1}, y_{j_1}) = (y_i, x_i)$ *and* $(x_{j_2}, y_{j_2}) = (1-x_i, 1-y_i)$, *then*

$$D_N^*(|x_j - y_j|) \leqslant 3D_N((x_j, \{y_j - x_j\})) + D_N((x_j, y_j)). \tag{10}$$

Proof. Basic idea used from [Z2]. As in [KN] we define the counting function

$$A(I; N; z_j) = \#\{j \leqslant N; z_j \in I\}.$$

Clearly, by Fig. 1,

$$A([0,x); N; |x_j - y_j|) = A(C \cup D; N; (x_j, y_j))$$

and thus, to estimate discrepancies we must transform the convex region $C \cup D$ to a rectangle. To do this we shall use a transformation $\psi : [0,1]^2 \to [0,1]^2$ defined by

$$\psi(x,y) = (x, \{y - x\}),$$

where $\{y - x\}$ is the fractional part of $y - x$.

(1) Assume first that $x \leqslant \frac{1}{2}$. Then $[0,1]^2$ is transformed by ψ as in Fig. 2 (we gave the diagonal $x = y$ in C). Divide the convex region $C \cup D$ to $C_1 \cup C_2 \cup C_3 \cup D_1 \cup D_2 \cup D_3$ and then apply the transform ψ (cf. Fig. 3). We see that

- $A(C_1 \cup C_2; N; (x_j, y_j)) = A(\psi(C_1 \cup C_2); N; (x_j, \{y_j - x_j\}))$,
- $A(D_2 \cup D_3; N; (x_j, y_j)) = A(\psi(D_2 \cup D_3); N; (x_j, \{y_j - x_j\}))$,
- $|C_1 \cup C_2| = |\psi(C_1 \cup C_2)| = x(1-x)$,
- $|D_2 \cup D_3| = |\psi(D_2 \cup D_3)| = x(1-x)$.

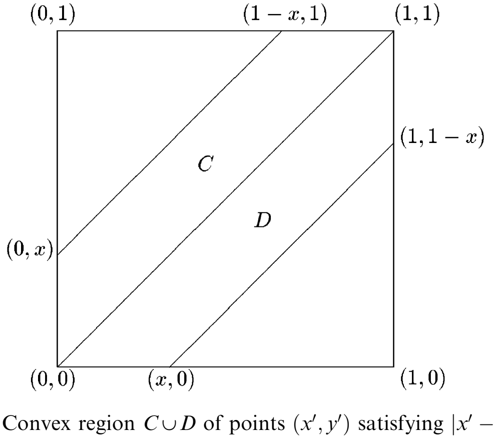

Fig. 1. Convex region $C \cup D$ of points (x', y') satisfying $|x' - y'| < x$.

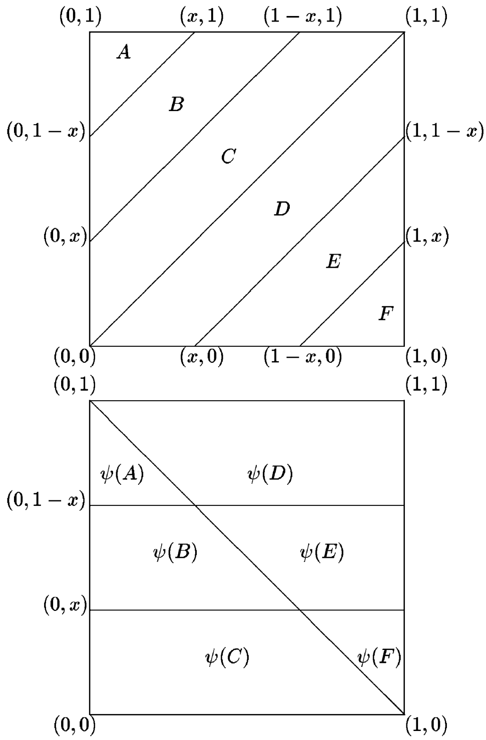

Fig. 2. ψ-Mapping of $[0,1]^2$.

Triangles C_3 and D_3 form a square and using transformations $(x,y)\to(y,x)$ and $(x,y)\to(1-x,1-y)$ we have

$$\left|\frac{A(C_3\cup D_3;N;(x_j,y_j))}{N}-|C_3\cup D_3|\right|=\left|\frac{A(C_3;N;(x_j,y_j))}{N}-|C_3|\right.$$
$$\left.+\frac{A(D_1;N;(x_j,y_j))}{N}-|D_1|\right|.$$

Since

- $A([0,x);N;|x_j-y_j|)=A(C\cup D;N;(x_j,y_j))$,
- $|C\cup D|=2x-x^2$,
- $C\cup D=C_1\cup C_2\cup C_3\cup D_1\cup D_2\cup D_3$

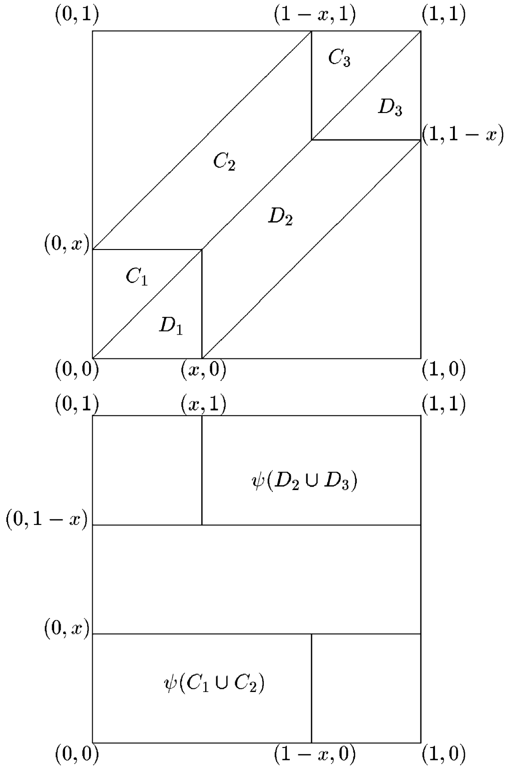

Fig. 3. ψ-Mapping of convex region from Fig. 1.

and applying the transformation ψ we have

$$\begin{aligned}
&\left|\frac{A([0,x);N;|x_j-y_j|)}{N}-(2x-x^2)\right|\\
&\quad=\left|\frac{A(\psi(C_1\cup C_2);N;(x_j,\{y_j-x_j\}))}{N}-|\psi(C_1\cup C_2)|+\frac{A(C_3;N;(x_j,y_j))}{N}-|C_3|\right.\\
&\qquad\left.+\frac{A(\psi(D_2\cup D_3);N;(x_j,\{y_j-x_j\}))}{N}-|\psi(D_2\cup D_3)|+\frac{A(D_1;N;(x_j,y_j))}{N}-|D_1|\right|\\
&\quad\leqslant 2D_N((x_j,\{y_j-x_j\}))+D_N((x_j,y_j))
\end{aligned}$$

for $x\leqslant\frac{1}{2}$.

(2) Now assume $x>\frac{1}{2}$. Then the points $(x',y')\in[0,1]^2$ satisfying $|x'-y'|<x$ form a convex set $B\cup C\cup D\cup E$ as in Fig. 4. and since by Fig. 2 $\psi(B\cup E)$ is a rectangle, we

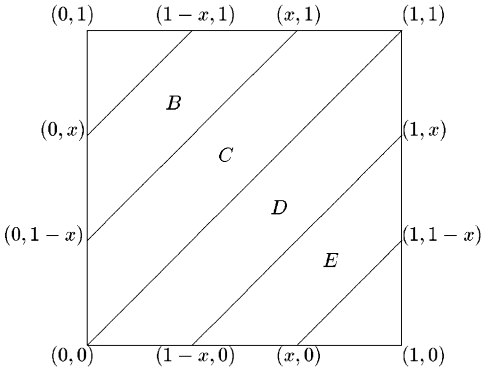

Fig. 4. Decomposition of convex region of Fig. 1 for $x > 1/2$.

obtain again

$$\left|\frac{A([0,x);N;|x_j-y_j|)}{N}-(2x-x^2)\right|$$
$$\leqslant 2D_N((x_j,\{y_j-x_j\}))+D_N((x_j,y_j))+D_N((x_j,\{y_j-x_j\})). \qquad \square$$

Note that the invariance of (x_j, y_j), $j = 1, 2, \ldots, N$, with respect to $(x, y) \to (1-x, 1-y)$ can be replaced by $x_j \neq y_j$ for $j = 1, 2, \ldots, N$.

6. Discrepancy of B_n

Applying Erdös–Turán–Koksma inequality (3) to the sequence $(x_j, \{y_j - x_j\})$, $j = 1, 2, \ldots, N$ we have

$$D_N((x_j,\{y_j-x_j\})) \leqslant \left(\frac{3}{2}\right)^2 \left(\frac{2}{H+1}+\sum_{0<||h||\leqslant H}\frac{1}{r(\mathbf{h})}\left|\frac{1}{N}\sum_{j=1}^{N} e^{2\pi i((h_1-h_2)x_j+h_2y_j)}\right|\right).$$

Then putting

$$(x_j, y_j) = \left(\frac{a_j}{n}, \frac{a_j^*}{n}\right), \quad N = \varphi(n),$$

and since for g.c.d. $(h_1 - h_2, h_2, n) = (h_1, h_2, n)$, we find

$$D_{\varphi(n)}\left(\left(\frac{a_j}{n},\left\{\frac{a_j^*}{n}-\frac{a_j}{n}\right\}\right)\right) \leqslant \left(\frac{3}{2}\right)^2 \left(\frac{2}{H+1}+\frac{d(n)\sqrt{n}}{\varphi(n)}\sum_{0<||h||\leqslant H}\frac{\sqrt{(h_1,h_2,n)}}{r(\mathbf{h})}\right).$$

Here

$$\sum_{0<||h||\leqslant H} \frac{\sqrt{(h_1,h_2,n)}}{r(\mathbf{h})} \leqslant 4 \sum_{h_1,h_2=1}^{H} \frac{\sqrt{(h_1,h_2,n)}}{h_1 h_2} + 4 \sum_{h_1=1}^{H} \frac{\sqrt{(h_1,n)}}{h_1}.$$

Similarly as in Part 3, for the first sum we use the upper bound $3\,(\log(eH))^2$ and for the second sum we use

$$\sum_{h_1=1}^{H} \frac{\sqrt{(h_1,n)}}{h_1} \leqslant \sum_{d|n} \frac{1}{\sqrt{d}} \sum_{h=1}^{H} \frac{1}{h} \leqslant d(n)\log(eH),$$

and thus

$$\begin{aligned} &D_{\varphi(n)}\left(\left(\frac{a_j}{n}, \left\{\frac{a_j^*}{n} - \frac{a_j}{n}\right\}\right)\right) \\ &\leqslant \left(\frac{3}{2}\right)^2 \left(\frac{2}{H+1} + 12\,\frac{d(n)\sqrt{n}}{\varphi(n)}(\log(eH))^2 + 4\,\frac{d^2(n)\sqrt{n}}{\varphi(n)}\log(eH)\right) \end{aligned}$$

which for $eH = \varphi(n)$ gives

$$D_{\varphi(n)}\left(\left(\frac{a_j}{n}, \left\{\frac{a_j^*}{n} - \frac{a_j}{n}\right\}\right)\right) \leqslant \left(\frac{3}{2}\right)^2 17\,\frac{d^2(n)\sqrt{n}}{\varphi(n)}(\log\varphi(n))^2$$

and then by (10) we find

$$D_{\varphi(n)}\left(\left|\frac{a_j}{n} - \frac{a_j^*}{n}\right|\right) \leqslant 4\left(\frac{3}{2}\right)^2 17\,\frac{d^2(n)\sqrt{n}}{\varphi(n)}(\log\varphi(n))^2$$

for $n \geqslant 8$.

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